DUAL STRUCTURES IN NON-COMMUTATIVE DIFFERENTIAL ALGEBRAS

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Abstract

The non-commutative algebraic analog of the moduli of vector and covector fields is built. The structure of moduli of derivations of non-commutative algebras are studied. The canonical coupling is introduced and the conditions for appropriate moduli to be reflexive are obtained.

FOREWORD

The duality problem we are going to tackle stems from the non-commutative generalization of differential geometry of manifolds. The variety of structures it deals with is defined in terms of two basic objects: the algebra $\mathcal{A}$ of smooth functions on a manifold and the Lie algebra $\mathcal{V}$ of smooth vector fields. In standard differential geometry $\mathcal{V}$ is always a *reflexive* $\mathcal{A}$-module which enables the tensor analysis to be successively built. Whereas it is worthy to note that the fundamental geometrical notions are formulated in pure algebraic terms of *commutative* algebra $\mathcal{A}$ [2]. The attempts of direct generalization of these notions starting from a non-commutative algebra $\mathcal{A}$ produce non-trivial algebraic problems: we focus on two of them.

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The first is to introduce an analog of the module of vector fields: in classical geometry it is the $\mathcal{A}$-module $\mathcal{V} = \mathsf{Der}\mathcal{A}$, meanwhile when $\mathcal{A}$ is non-commutative $\mathsf{Der}\mathcal{A}$ is not $\mathcal{A}$-module. To overcome it, we have to restrict $\mathsf{Der}\mathcal{A}$ to some its Lie subalgebra $\mathcal{V}$, and introduce the set of scalars $\mathcal{S}$ *with respect to* $\mathcal{V}$. In this paper we shall use the center $\mathcal{Z}$ of $\mathcal{A}$ as the set of scalars.

The second problem is to introduce an appropriate definition of the dual module of $\mathcal{V}$. The matter is that the standard dualization borrowed from the theory of moduli turns out to be incompatible with the generalization of forthcoming geometrical notions such as, for instance, Cartan differential. The guideline to solve the second problem is to choose the dual module so that the obtained pair of dual objects would be reflexive. The standard definitions related with rings and moduli are borrowed from [1].

The paper is structured as follows. In section 1 the notion of differential algebra is introduced as the couple (associative algebra, module of vector fields). In section 2 different ways to build the dual to the module of 'vector fields' are discussed and the choice of appropraite definition is motivated. In section 3 the conditions for the module of vector fields to be refleive are introduced.

1 DIFFERENTIAL ALGEBRAS

Let $\mathbf{F}$ be a field with zero characteristic, and consider a (non-commutative, in general) associative algebra with the unit element over $\mathbf{F}$. Denote by $\mathsf{Der}\mathcal{A}$ the set of the set of DERIVATIVES of $\mathcal{A}$, that is the $\mathbf{F}$-linear mappings $v : \mathcal{A} \longrightarrow \mathcal{A}$ which enjoy the Leibniz rule: for any $a, b \in \mathcal{A}$

$$v(ab) = va \cdot b + a \cdot vb$$

$\mathsf{Der}\mathcal{A}$ is the Lie algebra over $\mathbf{F}$ with respect to the bracket operation $[u, v]a = uv(a) - (vu)a$. The action of $\mathsf{Der}\mathcal{A}$ on $\mathcal{A}$ induces the Galois connection between subsets of $\mathcal{A}$ and $\mathsf{Der}\mathcal{A}$ defined as follows. For each $\mathcal{B} \subseteq \mathcal{A}$ ($\mathcal{V} \subseteq \mathsf{Der}\mathcal{A}$, resp.) define the polar $\mathcal{B}^c$ ($\mathcal{V}^c$, resp.) as:

$$\mathcal{B}^c = \{v \in \mathcal{V} \mid \forall b \in \mathcal{B} \quad vb = 0\} \qquad \mathcal{V}^c = \{a \in \mathcal{A} \mid \forall v \in \mathcal{V} \quad va = 0\} \tag{1}$$

It follows from the general theory of Galois connections that $\mathcal{V}^{ccc} = \mathcal{V}^c$ and $\mathcal{B}^{ccc} = \mathcal{B}^c$.

Proposition 1 (i). *For any $\mathcal{V} \subseteq \mathsf{Der}\mathcal{A}$ the set $\mathcal{V}^c$ is the subalgebra of $\mathcal{A}$.*

(ii). *For any $\mathcal{B} \subseteq \mathcal{A}$ the set $\mathcal{B}^c$ is the Lie subalgebra of* $\mathsf{Der}\mathcal{A}$.

Proof is yielded by direct verification of appropriate definitions. $\square$

For any $a \in \mathcal{A}$, $v \in \mathcal{V}$ the linear mapping $av : \mathcal{A} \longrightarrow \mathcal{A}$ is defined as

$$(av)b = a \cdot vb \tag{2}$$

Definition 1 *A* DIFFERENTIAL ALGEBRA *is a pair $(\mathcal{A}, \mathcal{V})$ where $\mathcal{A}$ is an algebra and $\mathcal{V} \subseteq \mathsf{Der}\mathcal{A}$ are such that $\mathcal{V} = \mathcal{V}^{cc}$.*

It follows immediately from Proposition 1 that $\mathcal{V}$ is always the Lie subalgebra of $\mathsf{Der}\mathcal{A}$. There are two algebras associated with $(\mathcal{A}, \mathcal{V})$:

$$\mathbf{C} \ = \ \mathcal{V}^c \qquad \text{the algebra of constants} \tag{3}$$

It turns out that a differential algebra can be equivalently defined as the pair $(\mathcal{A}, \mathbf{C})$, with $\mathbf{C} = \mathbf{C}^{cc} \subseteq \mathcal{A}$, which coincides with the above definition 1 by putting $\mathcal{V} = \mathbf{C}^c$. In any differential algebra $\mathcal{V}$ possesses the natural structure of the module over the commutative algebra $\mathcal{Z}$ (the center of $\mathcal{A}$).

2 DUALITY

There is the *canonical* construction of conjugated to a left (right) $\mathcal{Z}$-module which is the right (resp., left) $\mathcal{Z}$-module. So, the conjugated to the left $\mathcal{Z}$-module $\mathcal{V}$ will be the right $\mathcal{Z}$-module

$$\mathcal{V}^* = \hom_{\mathcal{Z}}(\mathcal{V}, \mathcal{Z})$$

Likewise the left $\mathcal{Z}$-module $\mathcal{V}^{**}$ is introduced. There is the canonical homomorphism $\mathcal{V} \longrightarrow \mathcal{V}^{**}$ of left $\mathcal{Z}$-moduli which is not isomorphism in general. The reflexive moduli (with $\mathcal{V} = \mathcal{V}^{**}$) are normally "good" objects to deal with. In our paper we are going to fetch the reflexivity to the construction by appropriate choice of the definition of conjugation taking into account the features of $\mathcal{V} \subseteq \mathsf{Der}\mathcal{A}$. Consider the set:

$$\mathcal{V}^+ = \hom_{\mathcal{Z}}(\mathcal{V}, \mathcal{A})$$

of all Z-linear forms on $\mathcal{V}$ taking values in $\mathcal{A}$ (rather than in $\mathcal{Z}$). Call the elements of $\mathcal{V}$ vectors and the elements of $\mathcal{V}^+$ covectors. The following propositions show the relevance of such definition.

Proposition 2 *$\mathcal{V}^+$ possesses the natural structure of $\mathcal{A}$-bimodule.*

Proof The additive structure on $\mathcal{V}^+$ is introduced in the standard way, and the right action of the elements of $\mathcal{A}$ is defined as:

$$(\omega \cdot a)(v) = \omega(v)a \qquad \omega \in \mathcal{V}^+ \quad , \quad a \in A, v \in V$$

making $\mathcal{V}^+$ right $\mathcal{A}$-module. For any $a \in \mathcal{A}$, $\omega \in \mathcal{V}$ consider the $\mathcal{A}$-valued function $(a\omega)(v) = a\omega(v)$ on $\mathcal{V}$. To check that $a\omega \in \mathcal{V}^+$, it suffices to check its $\mathcal{Z}$-linearity. For each $s \in \mathcal{Z}$ consider the discrepancy $\delta = a\omega(sv) - sa\omega(v) = [a,s]\omega(v) = 0$, since $s \in \mathcal{Z}$. Finally, for each $v \in \mathcal{V}$ we have $(a\omega(v))b = a\omega(v)b = a(\omega(v)b)$, hence

$$(a\omega)b = a(\omega b) \qquad a, b \in \mathcal{A} \quad , \quad \omega \in \mathcal{V}^+$$

therefore $\mathcal{V}^+$ is $\mathcal{A}$-bimodule. □

Remarks. (i). This unexpected feature of $\mathcal{V}^+$ was enabled by that $\mathcal{V}$ is the module of derivations.
(ii). In general $a\omega \neq \omega a$.

Each element $a \in \mathcal{A}$ canonically induces the $\mathbf{F}$-linear mapping $da : \mathcal{V} \longrightarrow \mathcal{A}$ defined as $da(v) = va$. The next statement solves the problem of generalization of Cartan's differentials.

Proposition 3 (i). *$da \in \mathcal{V}^+$ for any $a \in \mathcal{A}$.*

(ii). $d(ab) = da \cdot b + a \cdot db$

(iii). $\ker d \quad = \quad \mathbf{C} \quad = \quad \mathcal{V}^c$

Proof (i) $da(sv) = sva = sda(v)$, (ii) is verified by direct checking, (iii) follows directly from the definitions (1,3). $\square$

We shall call the elements of $\mathcal{V}^+$ of the form da DIFFERENTIALS. Consider the $\mathcal{A}$-bisubmodule of $\mathcal{V}$ generated by differentials:

$$\mathbf{\Omega} = \{\omega \in \mathcal{V}^+ \mid \omega = \sum_i \rho_i da_i q_i\}$$

where i ranges over a finite set and call $\mathbf{\Omega}$ the MODULE OF DIFFERENTIAL FORMS. Introduce the set

$$\mathbf{W} = \hom_{\mathcal{A},\mathcal{A}}(\mathcal{V}^+, \mathcal{A}) \tag{4}$$

of all $\mathcal{A}$-bilinear homomorphisms $\mathcal{V}^+ \longrightarrow \mathcal{A}$.

Proposition 4 $\mathbf{W}$ *is* $\mathcal{Z}$*-bimodule.*

Proof For $w \in \mathbf{W}$, $s \in \mathcal{Z}$ define the mapping $sw : \mathcal{V}^+ \longrightarrow \mathcal{A}$ as $(sw)\omega = s \cdot w\omega$. In the similar way the right action of $\mathcal{Z}$ is introduced. Then prove that swt is $\mathcal{A}$-bilinear for any $s, t \in \mathcal{Z}$:

$$(swt)(a\omega b) = s \cdot a \cdot w\omega \cdot b = a \cdot (sw\omega t) \cdot b$$

$\square$

The bilinear form $< \cdot, \cdot >$ on $\mathcal{V} \times \mathcal{V}^+$ with the values in $\mathcal{A}$ naturally arises:

$$< v, \omega >= \omega(v)$$

Note that no confusion occurs: $< sv, a\omega b >= sa < v, \omega b >= as < v, \omega > b$ for $s \in \mathcal{Z}$. This bilinear form enables the canonical homomorphism $j : \mathcal{V} \longrightarrow \mathbf{W}$ defined as:

$$jv =< v, \cdot > \tag{5}$$

In the category of moduli there always exists the natural homomorphism $\mathcal{V} \longrightarrow \mathcal{V}^{**}$, which is in general neither isomorphism, nor even embedding. In our scheme the rôle of $\mathcal{V}^{**}$ is played by $\mathbf{W}$, and the following holds:

Proposition 5 *The homomorphism* j *embeds* $\mathcal{V}$ *into* $\mathbf{W}$ *as direct summand.*

Proof For each $w \in \mathbf{W}$ define the mapping $\hat{w} : \mathcal{A} \longrightarrow \mathcal{A}$ as follows: $\hat{w}(a) = w(da)$. Then $\hat{w}(ab) = w(da \cdot b) + w(a \cdot db) = wa \cdot b + a \cdot \hat{w}b$, hence $\hat{w} \in \mathsf{Der}\mathcal{A}$. Since $\hat{w}(k) = w(dk) = 0$ for any $k \in \mathbf{C}$, $w \in \mathcal{V}$ (it follows from the definition of differential algebra that $\mathcal{V} = \mathbf{C}^c$). Denote $w \mapsto \hat{w}$ by π:

$$\pi(w) = \hat{w}$$

Evidently, π is $\mathcal{Z}$-linear, therefore it is the homomorphism $\mathbf{W} \longrightarrow \mathcal{V}$. Then consider $(\pi j v)(a) =< v, da >= va$, hence $\pi j = 1_{\mathcal{V}}$, thus j is the embedding. $\square$

Proposition 6 *Consider two submoduli of* $\mathbf{W}$*:* $\mathsf{Im} j$ *(being isomorphic to* $\mathcal{V}$*) and* $N = \mathsf{Ker}\pi$*. Then*

$$\mathbf{W} = \mathcal{V} \oplus N \tag{6}$$

Proof Consider the endomorphism $j_\pi = j \circ \pi$ of $\mathbf{W}$. Then $\mathsf{Ker} j_\pi = \mathsf{Ker}\pi = N$ (since $\mathsf{Ker} j = 0$), and $\mathsf{Im} j_\pi = \mathsf{Im} j$ (since $\mathsf{Im}\pi = \mathcal{V}$). Finally, ${j_\pi}^2 = \pi j \pi j = j_\pi$, hence $\mathbf{W} = \mathsf{Im} j \oplus N$. $\square$

Since j is the embedding, in the sequel we shall identify $\mathcal{V}$ with its image, so

Corollary. N is exhausted by annihilators of differential forms in $\mathcal{V}^+$:

$$N = \mathsf{Ann}\boldsymbol{\Omega} = \{n \in \mathbf{W} \mid \forall \omega \in \boldsymbol{\Omega} \quad \omega(n) = 0\} \tag{7}$$

3 REGULARITY AND REFLEXIVITY

Consider an important special case.

Proposition 7 *If* $\mathcal{V}$ *is the finitely generated free module, then* $\mathcal{V}$ *is canonically isomorphic to* $\mathbf{W}$*.*

Proof Recall that $\mathcal{Z}$ is a subalgebra of $\mathcal{A}$, hence

$$\mathcal{V}^* = \hom_s(\mathcal{V}, \mathcal{S}) \subseteq \mathcal{V}^+$$

Let $v_1, \ldots, v_n$ be a basis of $\mathcal{V}$. Build its dual basis $\omega^1, \ldots, \omega^n$ in $\mathcal{V}^*$: $\omega^i(v_k) = \delta^i_k$. Note that $\omega^i(v) \in \mathcal{S}$ and prove that $\{\omega^i\}$ is the basis of $\mathcal{V}^+$. For any $\omega \in \mathcal{V}^+$ we have $\omega(v) = \omega^i(\sum \omega(v) \cdot v_i) = \sum \omega^i(v) \cdot \omega(v_i)$, hence

$$\omega^i = \sum \omega \cdot \omega(v_i) \tag{8}$$

Moreover, for any $a \in \mathcal{A}$, $a\omega^i(v_k) = a\delta^i_k = \delta^i_k a = (\omega^i a)(v_k)$, hence

$$a\omega^i = \omega^i a \tag{9}$$

therefore $\forall w \in \mathbf{W}$ we have $aw(\omega^i) = w(\omega^i)a$, and thus $w(\omega^i) \in \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{S}$. Denote $w_i = j(v_i)$, and show that any $w \in \mathbf{W}$ can be decomposed over w_i: for any $\omega \in \mathcal{V}^+$ $\;w(\omega) = \sum w(\omega^i)\omega(v_i) =$

$$\sum \omega(w(\omega^i)v_i) = \omega(\sum w(\omega^i)v_i) =< \sum w(\omega^i)v_i, \omega >=$$

$$\sum w(\omega^i) < v_i, \omega >= \sum w(\omega^i) \cdot jv_i(\omega) = \sum w(\omega^i) \cdot w_i(\omega)$$

So, $\{w\}$ is the generating set for $\mathbf{W}$, hence $\mathbf{W}$ coincides with the image of the injection j, which completes the proof. □

Since $\mathbf{W}$ splits into the direct sum (6), we can also split out the module $\mathcal{V}^+$. Namely, consider the set R of annihilators of N:

$$R = \mathsf{Ann}N = \{\omega \in \mathcal{V}^+ \mid \forall n \in N \quad n(\omega) = 0\}$$

and call them REGULAR COVECTORS. R is the double annihilator of the module of differential forms:

$$R = \mathsf{Ann}(\mathsf{Ann}\boldsymbol{\Omega}) \subseteq \mathcal{V}^+$$

A differential algebra $(\mathcal{A}, \mathcal{V})$ is said to be REFLEXIVE if its set of covectors is closed: $\mathcal{V}^+ = R$. The adequacy of using the term 'reflexive' is corroborated by the following

Proposition 8 *The following two conditions are equivalent:*

(i). *$(\mathcal{A}, \mathcal{V})$ is reflexive.*

(ii). *$\mathbf{W} = \mathcal{V}$.*

Proof is straightforward. $\square$

Remark. It follows from Proposition 7 that any differential algebra with a free finitely generated module of vectors is always reflexive. However, the converse is not true: the counterexample is yielded by the algebra $\mathcal{A}$ of smooth functions on the 2-dimensional sphere with $\mathcal{V} = \mathsf{Der}\mathcal{A}$ (since each covector is differential form).

Denote by R^+ the dual to R:

$$R^+ = \hom_{\mathcal{A},\mathcal{A}}(R, \mathcal{A}) \tag{10}$$

Since $R \subseteq \mathcal{V}^+$ and $\mathbf{W} = \hom_{\mathcal{A},\mathcal{A}}(\mathcal{V}^+, \mathcal{A})$, there is the natural homomorphism $\beta : \mathbf{W} \longrightarrow R^+$ such that:

$$\beta(w) = w\mid_R$$

for any $w \in \mathbf{W}$ (called the restriction homomorphism).

Proposition 9 $\mathsf{Ker}\beta = N$.

Proof $\mathsf{Ker}\beta = \{w \in \mathbf{W} \mid \forall \omega \in R w(\omega) = 0\} = \mathsf{Ann}R = \mathsf{Ann}(\mathsf{Ann}(\mathsf{Ann}\boldsymbol{\Omega})) = \mathsf{Ann}\boldsymbol{\Omega} = N$ by virtue of (7). $\square$

Corollaries. (i). The homomorphism β can be canonically decomposed into canonical projection j_π and monomorphism i:

$$\begin{tikzcd} \mathbf{W} \arrow[rr, "\beta"] \arrow[dr, "\rho"'] & & R^+ \\ & \mathbf{W}/N \arrow[ur, "i"'] & \end{tikzcd} \tag{11}$$

(ii). $\mathcal{V}^+ = R$ whenever β is epimorphism.
(iii). $\mathcal{V}$ is the submodule of R^+ (since $\mathbf{W}/N = \mathcal{V}$). Moreover, the following holds:

Proposition 10 *$\mathcal{V}$ is the direct summand of R^+.*

Proof We already have the injection $i : \mathcal{V} \longrightarrow R^+$ (11). To complete the proof, it suffices to build the projection $\pi : R^+ \longrightarrow \mathcal{V}$ such that $\pi i = id_{\mathcal{V}}$. Define for $x \in R^+$ the mapping $\pi x : \mathcal{A} \longrightarrow \mathcal{A}$

$$\pi x(a) = x(da)$$

Like in the Proposition 5 it can be proved that $\pi x \in \mathcal{V}$. Now for $v \in \mathcal{V}$ consider $\pi(iv)(a) = iv(da) = \beta jv(da) = jv(da)$ since β is the restriction homomorphism, hence $\pi(iv)(a) = jv(da) = v(a)$ by the definition of the mapping $j : \mathcal{V} \longrightarrow \mathbf{W}$ (5). □

Proposition 11 *If $\mathcal{A}$ considered bimodule over $\mathcal{A}$ itself is injective then $\mathcal{V}^+ = R$.*

Proof It suffices to prove that the injection i is surjective. $\mathcal{A}$ is injective, therefore the exactness of the sequence of the $\mathcal{A}$-bimoduli

$$0 \longrightarrow R \xrightarrow{\text{in}} \mathcal{V}^+$$

implies the exactness of

$$\hom_{\mathcal{A},\mathcal{A}}(\mathcal{V}^+, \mathcal{A}) \longrightarrow \hom_{\mathcal{A},\mathcal{A}}(R, \mathcal{A}) \longrightarrow 0$$

The first term of the latter sequence is $\mathbf{W}$ (4), and the second is R^+ (10). The homomorphism between them induced by the inclusion $\mathsf{in} : R \longrightarrow \mathcal{V}^+$ is just the restriction homomorphism β, so the sequence

$$\mathbf{W} \xrightarrow{\beta} R^+ \longrightarrow 0$$

is exact which means that β is surjection. Finally, the commutativity of the diagram (11) makes i be the surjection. □

AFTERWORD

Two problems concerning the non-commutative generalization of differential structure were put forward in the Foreword.

The first was to suggest a suitable choice of subalgebras $\mathcal{V}$ of $\mathsf{Der}\mathcal{A}$ playing the role of vector fields. We did it by introducing the closed subalgebras with respect to constants and scalars (section 1).

Section 2 prepares the necessary technical tools to form the pair of dual objects. Several candidates for being dual objects are suggested and some results concerning their structure (such as direct decomposability) are obtained.

In section 3 the notion of reflexive differential algebra $(\mathcal{A}, \mathcal{V})$ is introduced for which the following condition holds:

$$\mathcal{V} = \hom(\hom(\mathcal{V}, \mathcal{A}), \mathcal{A})$$

where hom is the set of the morphisms in the aproprate category.

The following differential algebras $(\mathcal{A}, \mathcal{V})$ were proved to be reflexive whenever *at least one* of the following conditions holds:

- $\mathcal{V}$ is finitely generated free module
- $\mathcal{A}$ is injective $\mathcal{A}$-bimodule

The obtained results are applied to the problem of quantization of gravity [3]. The algebraic account of this problem was first done by Geroch [2] where the commutative case was considered. From the general algebraic point of view the idea to vary the definition of dual object looks rather prospective. In particular, a representation theory for general partially ordered sets was built along these lines [5, 4].

References

[1] Faith, C., Algebra: Rings, Modules and Categories, I, Springer, 1973
Faith, C., Algebra II: Ring Theory, Springer, 1976

[2] Geroch, R., Einstein Algebras, Communications in Mathematical Physics, **26**, 271, 1972

[3] Parfionov, G.N., Zapatrin, R.R., *Pointless Spaces in General Relativity*, International Journal of Theoretical Physics, **34**, 775, 1995

[4] Zapatrin, R.R., *Les espaces duaux pour les ensembles ordonnés*, Comptes Rendus de l'Académie des Sciences de Paris, ser. Mathematiques, submitted in 1995

[5] Zapatrin, R.R., *Algebraic duality in the theory of partially ordered sets*, Order, submitted in 1996